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Learning trajectory of non-Euclidean geometry through ethnomathematics learning approaches to improve spatial ability

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Abstract

Non-Euclidean geometry is an abstract subject and difficult to learn, but mandatory for students. The ethnomathematics approach as a learning approach to improve students' spatial abilities. The aim of this research is to discover new elements of the spatial abilities of non-Euclidean geometry; determine the relationship between spatial abilities for Euclid, Lobachevsky, and Riemann geometry. This study used the micro genetic method with a 2×2 factorial experimental research design. The sample of this research is 100 students of mathematics education. There are three valid and reliable research instruments through expert trials and field trials. Data collection was carried out in two ways, namely tests and observations. Quantitative data were analyzed through ANCOVA, and observational data were analyzed through the percentage of implementation of the learning trajectory stages. The result is that the spatial ability of students who are given the ethnomathematics learning approach is higher than students who are given the conventional learning approach for Lobachevsky geometry material after controlling for the effect of Euclidean geometry spatial ability. Also, the same thing happened for the spatial abilities of Riemann geometry students. The learning trajectory is conveying learning objectives (learning objective); providing ethnomathematics-based visual problems; students do exploration; students make conclusions and summaries of exploration results; and ends with students sharing conclusions/summaries about concepts and principles in geometric systems. It was concluded that learning non-Euclid geometry through learning paths with an ethnomathematics approach had a positive impact on increasing students' spatial abilities.

Keywords: learning trajectory, spatial ability, geometry, non-Euclid, ethnomathematics

INTRODUCTION

Geometry learning is very important for students who are prospective mathematics teachers. That is because in schools teachers are needed who are able to teach Euclid and non-Euclid geometry. However, there are still many mathematics education students who will teach in schools have difficulty understanding the concepts and principles of geometry, especially non-Euclid. Therefore, an ethnomathematics approach is needed to make it easier to understand. Spatial ability is an important competence in understanding geometry. However, the elements that exist to date are more elements of the spatial capabilities of Euclid geometry (Maier, 1998), so new spatial elements are needed for non-Euclid geometry. To achieve this requires the genetic decomposition of the research subject through the framework of theory of action-proses-object-scheme (APOS) (Dubinsky & McDonald, 2000).

Learning geometry with a real-world approach, broad concepts can be extended to other forms. Such as connecting 'vast' with other 'big'; investigating the relationship between area and circumference; connecting the unit of measurement with reality; also, integrating some geometric activities (Fauzan et al., 2002). In the learning of geometry (in general mathematics), the relationship with reality becomes a significant process. It is emerging from the mathematical

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Contribution to the literature

- This study found a new theory about learning trajectories to improve the spatial abilities of non-Euclidean geometry, namely, understanding learning objectives, understand visual problems based on ethnomathematics, do exploration, students make conclusions and summaries of exploration results, and ends with students sharing summaries.
- The findings show that there is a trajectory of students' spatial thinking in understanding non-Euclidean geometry through an ethnomathematical learning approach in terms of APOS theory.
- The results showed that there was an effect of the ethnomathematical approach on the students' spatial ability in non-Euclidean geometry learning.

reality. Like the idea from Freudenthal that reality as a framework is inherent in mathematics itself (Gravemeijer, 2008; Plomp & Nieveen, 2013). During geometry learning, we can interpret the mathematical process of students in solving problems or in an effort to achieve certain concepts or principles. The cognitive process is a mathematical process that can be analyzed through the genetic decomposition of students (Cooley et al., 2007; Widada et al., 2019a, 2020). Genetic decomposition is a structured collection of mental activities that a person undertakes to describe how mathematical concepts/principles can be developed in his mind (Cooley et al., 2007; Widada, 2002). Genetic decomposition analysis is an analysis of a genetic decomposition based on the activity of actions, processes, objects, and schemes (APOS theory) carried out by a person in mathematical activity (Widada, 2017).

Based on Maier (1998) that the elements of spatial ability are spatial perception ability, visualization ability, mental rotation ability, spatial relation ability, and spatial orientation ability. Spatial perception ability is an ability that requires the location of the object being observed horizontally or vertically. Visualization ability is the ability to show the rules of change or movement of the constituents of a building either three-dimensional to two-dimensional or vice versa. Mental rotation ability is the ability to rotate two-dimensional and three-dimensional objects precisely and accurately. The ability of spatial relations is the ability to understand the arrangement of an object and its parts and their relationship with each other. The ability spatial orientation is the ability to observe an object from various circumstances (Maier, 1998).

Geometry is a compulsory subject for mathematics education students (Nugroho et al., 2019). Objects are abstract, so students often have difficulty understanding them (Widada et al., 2019b). Therefore, teachers must be able to manage geometry learning appropriately. It is carried out through reflection on the initial abilities of the student. Teachers should conduct needs analysis, concept analysis, and task analysis (Nugroho et al., 2019).

Geometry is a deductive structure in mathematics. The structure starts from a primitive element consisting of points, lines and planes. Axioms in geometry become base statements. It is a true statement without having to be proved in its structure (Frassia & Serpe, 2017). An axiom is a true statement that serves to avoid the swirling of proofs. In geometry, there is a sense that limits a concept. That is a definition. Statements that are logical consequences in deductive structures in geometry are a theorem. The theorem must be proved to be true in the structure. Therefore, students are obliged to have mature cognitive processes (Dubinsky & McDonald, 2000). In addition, the results of the research of Wu and Ma (2006) suggest that investigating why elementary school students have difficulty in quadrilaterals, and for advanced geometry students have difficulty understanding and proving the properties of Saccheri quadrilaterals (initial survey of researchers). Preliminary findings suggest that quadrilaterals, except for squares and rectangles, are rarely displayed in textbooks, and in their daily lives. Because students are psychological beings who are able to actively process information.

A geometric system is a mathematical structure constructed by the set of all points with the basic entities of points, lines and planes. The system is built on the axiom of incidence (Eves, 1972). Eves states that there are three geometric systems namely Euclid geometry, Lobachevsky geometry, and Riemann geometry. Euclid constructs geometry (*the elements*) by basing five axioms, five postulates and twenty-three definitions (Hitchman, 2018). Euclid's five axioms are, as follows.

1. Through two different points can be made exactly one line. Can always draw a line from one point to another.
2. Through three different points and allusions always can create infinite line segments of many on a line.
3. Can always paint a circle centered on a point with the radius of the specified line segment.
4. All the angles of the elbows to each other are equally large.
5. If a straight line intersects two straight lines and makes the sum of the angles in unilateral less than two right angles, those two lines if extended infinitely will converge on side, where both angles in unilaterally are less than two right angles.

The axiom of Euclid alignment (5th axiom) is the basic principle of building the Euclid geometry system, but the axiom is disputed, so that at the beginning of the 20th century, two new axioms appeared. First is the axiom of Lobachevsky's alignment. The axiom became the main principle for constructing Lobachevsky geometry. Second was the Riemann parallel axiom, which later became the basic principle of the Riemann geometry system. Next can be discussed about Lobachevsky's geometry spatial ability, Riemann's geometry spatial ability, and Euclid's geometry spatial ability.

Based on the foregoing, ethnomathematics presents mathematical concepts (geometry) from the school curriculum in such a way as to concepts related to the culture and daily experiences of students (Hitchman, 2018). It was intended to improve learners' ability to decipher meaningful relationships and deepen their understanding of mathematics. The ethnomathematics approach to the math curriculum is intended to make school mathematics more relevant and meaningful to students and improve the overall quality of their education. In this context, applying an ethnomathematics perspective in the school math curriculum helps develop students' intellectual, social, emotional, and political learning by using their unique cultural references to convey their knowledge, skills, and attitudes. This kind of curriculum provides a way for students to maintain their identity while being academically successful.

Until the 19th century, the view of geometry was based on Euclid's *The elements*. Geometry as an axiomatic deductive system. It was a science of mathematical properties with the basic concept of the set of all points.

Relationships between geometric objects, i.e., points, lines and planes are in Euclid space (Anonimous, 2010). The principles in geometry are theorems and other properties are the result of examining points, lines and planes, as well as physical space and their mathematical relationships. However, at the beginning of the 20th century many scientists studied the axioms of Euclid's alignment, among them Lobachevsky and Riemann. The result of their study was the emergence of a new axiom about the alignment of lines. Lobachevsky states that "through a point outside a line there are at least two lines parallel to that line." It is with this axiom that a hyperbolic system of geometry is built. It was one of the alternatives of Euclid geometry. The opinion differs from Riemann, that "there are no parallel lines"(Hitchman, 2018). Riemann-pun built the elliptical geometry. Therefore there are three spaces in the geometric system, namely parabolic space (Euclid geometry), hyperbolic space (Lobachevksy geometry), and elliptical space (Riemann geometry) (Clayton, 2010).

In the development of geometry, there is one very important definition, namely the concept of the Saccheri quadrilateral.

Definition: An ABCD quadrilateral with right base angles (D and C angles) and congruent sides (AD=BC) is called a Saccheri quadrilateral. The side opposite the base is the apex, and the angle formed by the sides and the apex is the angle of the apex (angle A is equal to angle B). DM'=M'C and AM=MB. (Ross, 2010). Therefore, the right learning trajectory is needed so that students are able to understand the concepts of non-Euclid geometry correctly.

Learning trajectory in geometry learning is the steps of student learning activities in understanding geometry. It requires good spatial ability. Students have a dynamic sense of spatial form (Panorkou & Greenstein, 2015). Students see the shape by doing *a template matching* activity and then trying to explain the form identified is similar to what is in its *long-term memory* (Solso, 1995).

The notion of dynamic in geometry can be utilized in the environment around students, which is related to which positively affects students' geometry abilities. Students have the potential to harness the power of seeing geometric properties by breaking away from their original shapes through a process of abstraction and providing a foundation for the formation of geometric conceptions, such a dynamic essence they give students the opportunity to connect broader geometric concepts (Panorkou & Greenstein, 2015). The student's geometric thinking trajectory will affect his learning *trajectory*.

Learning trajectory is also expected to overcome teacher difficulties in teaching mathematics (Bednarz & Proulx, 2003) and student learning difficulties. The difficulty of learning mathematics is often explained by referring to the gap between students' personal knowledge and the abstract formal mathematical knowledge that needs to be acquired (Gravemeijer, 2008). Based on a constructivist perspective, through learning, knowledge is built by students through a process between people towards within individuals (internalization in their cognitive processes). Gravemeijer (2008) states that students who have not built up the more sophisticated mathematical knowledge that must be learned, this more sophisticated mathematical knowledge, literally, does not exist. It can all be built through its *learning trajectory*.

Hypothetical learning trajectory (HLT) is a theoretical model for mathematical learning design. It consists of three components namely

(1) learning objectives,

(2) a series of learning tasks, and

(3) a hypothesized learning process.

Constructs can be applied to instructional units of varying lengths (for example, one lesson, a series of lessons, the learning of concepts over an extended period of time (Simon, 2014) . HLT is a student's learning trajectory so that the mathematical concepts learned by elasticity can be understood by students. It is a way to describe pedagogic and didactic aspects in mathematics learning (Arnellis et al., 2019). The pedagogic aspect, namely the relationship between educators and students, and didactics is the relationship between students and the material. HLT is made to anticipate what might happen, both the thought process of students who will get learning and things that will happen in the learning process. Therefore, the implementation of *learning trajectory* increases motivation, positive attitudes appear such as being active, happy, and enthusiastic students following the lesson. Also, the high-level thinking ability test scores of experimental class students were greater than the average scores of the control class after being given treatment. According to Pratiwi et al. (2020) that learning *trajectory* in learning geometry about three dimensions with a *rigorous mathematical thinking* approach with a qualitative thinking level can improve problem-solving skills well. *The learning trajectory* meets qualitative thinking level indicators, especially visualization and labeling indicators. The trajectory is

(1) the student performs the stages of describing the desired shape, giving a name, and thinking of a plan to be carried out to solve a given problem with appropriate,

(2) the student explains his understanding of the concept of the material being taught so that the student masters the material, and

(3) students understand the problems given regarding three dimensions, so that students can understand the material and achieve the set learning objectives.

Simon (2014) states that the purpose of HLT is to provide an empirically based model of pedagogical thought based on constructivist ideas. Pedagogy refers to all contributions to instructional interventions including those made by curriculum developers, material developers, and teachers. HLT construction provides a theoretical framework for researchers, teachers, and curriculum developers as they plan instruction for conceptual learning. The components of HLT are

(1) math learning objectives for students,

(2) math tasks to be used to improve student learning (activities that support the goals), and

(3) hypotheses about student learning (mathematical conjectures as a result of activity) (Simon, 2014).

According to Prediger et al. (2015), structuring HLT on students' cognitive development, i.e., *conceptual leaning trajectory level*. There are HLT levels, i.e., starting from situational level is that student is in context of a specific situation. Referential level is a model and strategy that refers to situation described in the problem. General level is a focus on mastering mathematics with strategies that refer to context. Formal level is working with conventional procedures and notation.

Gravemeijer (2008) states that the situational level is the stage in which students are expected to use their informal knowledge and intuitive strategies in the context of real problems. The referential level is the situational level of the 'model-of', i.e., the student is expected to give rise to a symbol or mathematical model that refers to the real-life situation of a given problem. The general level is the 'model-for' level, i.e., students develop models that can be used in different situations. Students are expected to identify patterns and relationships so that they can apply strategies to different situations. Finally, the formal level is that students use their experience with the previous three levels to do reasoning. In this level the student is ready to work with procedures, algorithms or notations in deductive geometric systems (in general mathematics). In Lobachevsky's geometry learning, activity as a 'model of' ethnomathematics-based real situations such as line alignment in *bubu* (Herawaty et al., 2020), triangular properties based on Riemann through *grapefruit* medium (Widada et al., 2020) and "model for" more common problems may include parallel lines based on *bubu* fishing gear, and triangles based on children's toys using grapefruit peel. Students who already have sufficient knowledge based on all models, will be able to build formal geometry (Euclid and non-Euclid). Ethnomathematics is an effective approach to learning mathematics (Sunzuma et al., 2021). Also, the ethnomathematics approach is to train students in conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (Abah et al., 2021).

In implementing HLT, proper implementation steps are needed. According to Maloney and Confrey (2013), there are eight components of application in mathematics learning, namely

(1) understanding problems and being diligent in solving them,

(2) reasoning abstractly and quantitatively,

(3) build worthy arguments and critique the reasoning of others,

(4) models with mathematics,

(5) use the right tools strategically,

(6) pay attention to precision,

(7) search and utilize structures, and

(8) search and express rules in repeated reasoning.

While the learning trajectory consists of levels of thinking; each of them is more sophisticated than the latter, which leads to the achievement of mathematical goals, and HLT can facilitate teaching and developmentally appropriate learning for all learners (Clements & Sarama, 2010). That means that the learner's development of thinking illustrates the distinctive path he follows in developing understanding and skills about geometry.

Therefore, in the practice of learning geometry, the initial step is that students are treated to real problems as a *starting-point* of learning. It is to trigger his cognitive processes so that the student's mathematical abilities can be recalled from his *long-term memory*, so to the potential about mathematical reasoning, mathematical connections, mathematical communication, and mathematical representations owned by students can be optimally utilized to solve problems that given. Thus, students will be able to achieve geometric structures formally through a process of abstraction, idealization and generalization.

In learning geometry, the objects are the facts, concepts, principles and skills of geometry. Related to this, according to Panorkou and Greenstein (2015), the concept of having elements, namely connected attributes, traits and invariants, and the role played in the process achieve this through learning about these concepts. Starting from a real problem, students are expected to be able to classify objects based on their attributes, then students are expected to distinguish between their attributes and traits, then students It is expected to be able to predict the effects of dynamic transformations by transforming mental representations of their geometric shapes.

The basic paradigm of HLT is constructivism aimed at the development of students' mathematical understanding. The results of research on the trajectory of learning geometry, especially about triangles carried out by Anwar and Rofiki (2018) showed that students found the condition of forming a triangle with a length of three sides. The steps are

(1) *starting-point* of the student's learning trajectory is to draw a line of three given side lengths,

(2) students examine two side lengths whether or not they can be connected to a line segment as a triangle,

(3) students use a ruler to determine all three lengths of sides that can form a triangle, and

(4) student makes a statement that the sum of each of the two sides of the triangle must be greater than the third side.

In the process, the teacher conditions that the student's learning trajectory can achieve the learning objectives appropriately.

HLT in geometry contains three aspects, namely

(1) spatial,

(2) shape of planes and spaces (solid), and

(3) visualization and representation.

Each of its descriptions is linked to examples of activities that are described, analyzed and discussed based on core insights. The most important elements of such a teaching-learning trajectory are demonstrated in schematic representations, including topics related to its geometric activity (Gravemeijer et al., 2016). Such as spatial ability in looking at the ocean from the top of a hill; constructions about making balls; and representations about depicting parallel lines. The domain of the learning trajectory is visible during the learning activities. Therefore, in order for the learning of geometry that is deductive-axiomatic and abstract in nature it must be made starting from a problem close to the student's mind, the It is much more concrete so that learners can easily connect geometric objects in their cognitive systems.

In the curriculum the topic of similarity usually includes several parts ranging from the geometry of everyday life to deductive geometry. Traditionally, the teaching of geometry has largely adopted the Euclidean axiom approach (Zhang & Wong, 2021). As is the case today, the content and elements of classical mathematics books have a great influence on the mathematics curriculum. According to Zhang and Wong (2021), learning of geometry, outside of the spatial sense, is all deductive-axiomatic. In learning, deductive reasoning and meticulousness are needed in the geometric system. So evidence and proof play an important role in learning geometry. However, students have difficulty proving the principles in geometry.

Fish (1996) states that a possible way to address the above problem is the introduction of non-Euclid geometry at the school level. It is very important to identify students who have prerequisite knowledge and skills. A number of interesting teaching strategies, such as debates, discussions, investigations, and oral and written presentations, can be used to introduce and develop content material. Therefore, to learn non-Euclid geometry, it is necessary that the starting point for learning must be close to the student's local mind and culture, namely the ethnomathematics approach (Nugroho et al., 2021). Also, to improve the mathematics education curriculum (Gebre et al., 2021). The approach is one that is closer to the student's mind and daily life. Thus, the purpose of this research is to find the learning trajectory of non-Euclid geometry through an ethnomathematics learning approach to improve the spatial ability of non-Euclid geometry students.

METHOD

Participants

The participants are students of mathematics and mathematics education at universities in the Bengkulu Province, Indonesia. We selected 100 students through a simple random technique. The selection is based on the initial characteristics for each group of geometric spatial abilities.

This research implements a mixed method between qualitative and quantitative. The focus of this research is to explore students' spatial abilities during non-Euclidean geometry learning. The learning is carried out through an ethnomathematical approach. Students' understanding of non-Euclidean geometry concepts was analyzed based on APOS mental activity. To comply with COVID-19 health protocol, the implementation of learning is carried out hybrid (online and off-line according to the provisions of the head of the research site). This study also has exploratory characteristics.

Instrument

There are three quantitative research instruments, namely the Lobachevsky geometry spatial ability test, the Riemann geometry spatial ability test, and the Euclid geometry spatial ability test (test for accompanying covariates/variables). The instrument for the implementation of the learning trajectory is the observation sheet for the implementation of the learning plan (LP).

The researcher used a spatial ability test instrument to explore quantitative and qualitative data. Quantitative assessment rubric with ratio scale data. Quantitative data is also used as the basis for determining the research subjects who will be interviewed in depth. Interviews were conducted during and after learning geometry through an ethnomathematical approach. This research instrument has been reviewed and validated by five experts, it has also been tested on 30 students. Based on the expert test (panelist) of the Lobachevsky geometric spatial ability test instrument it was obtained that the average validity index of Aiken's is 0.85 and each grain >0.80 with a Cronbach's alpha coefficient of 0.744. The test showed that all items of the Lobachevsky geometry spatial ability test were valid and reliable. The results of the Riemann geometry spatial ability test expert test that each item showed Aiken's validity index of more than 0.80 and the average was 0.84 with a Cronbach's alpha coefficient of 0.770. This test gives the meaning that experts agree that the instrument of the Riemann geometry spatial ability test is valid and reliable. The third instrument is the Euclid geometry spatial ability test. The panelists also agreed that each item was valid and reliable, with the average Aiken's validity index of 0.85 with a Cronbach's alpha coefficient of 0.770. Thus, these three research instruments are valid and reliable based on expert tests. Furthermore, all these instruments were tested on 100 mathematics education students at a university in Bengkulu, Indonesia. The data of the trial results were analyzed using the help of the Lisrel 8.8 program. Analysis of the test data of the Euclid geometry spatial ability test instrument obtained the result that each test item was valid, this was shown from the T-value of each item more than 1.96, with an average of 6.824. The instrument is also reliable with a Cronbach's alpha of 0.79. For each item of the instrument the spatial ability of Lobachevsky geometry is valid with a T-value of more than 1.96 and the average T-value is 5.695. Its Cronbach's alpha is 0.88, which means reliable. Also, the T-value of each item of the instrument's spatial ability of Riemann geometry >1.96, which means it is valid with the average T-value being 6.767 and the Cronbach's alpha of 0.89. Thus, these three research instruments are standard and feasible to be used for data collection.

Table 1. Experimental design (factorial 2×2)

Content of geometry	Learning approach: Ethnomathematics (A1)	Learning approach: Conventional (A2)
Lobachevsky (B1)	A1B1	A1B2
Riemann (B2)	A2B1	A2B2

Data Collection Techniques

Learning is carried out during using the zoom meeting media. Also, supported by social media *WhatsApp*, YouTube, and e-mail. Learning is carried out for twelve meetings in a period of eight weeks. After the lesson was finished, we conducted a spatial ability test via google form for all students out of 100 students. The test results were analyzed using the existing assessment rubric. However, to determine the four research subjects, a pretest was carried out. So that we can conduct in-depth interviews during the learning process. Interviews were conducted using *WhatsApp* media via video call. This in-depth interview was conducted to explore students' spatial abilities on non-Euclidean geometry. This interview was recorded to obtain complete and accurate data. In the process of collecting data, the researcher sent a google form link from the research instrument to students. The arguments in the reason column are made openly for students to explore. An open field that has certain answers according to the students' cognitive processes.

Experimental Research Design

To answer a specific research problem regarding the effect of the ethnomathematical approach on geometric spatial abilities, experimental research was conducted. The design used is a 2×2 factorial design, as shown in **Table 1**.

Information from **Table 1** is that A1 (ethnomathematics learning approach); A2 (conventional learning approach); B1 (Lobachevsky geometry material); B2 (material Riemann geometry). Meanwhile, X is Euclid's geometry spatial ability as a covariate; Y is the spatial ability of non-Euclidean geometry (Lobachevsky geometry and Riemann geometry). For the four cells in **Table 1**, A1B1 is the sample group taught by an ethnomathematics approach with Lobachevsky geometry; A1B2 is a sample group that is taught by an ethnomathematics approach with

Table 2. Non-Euclid geometry lecture activities

Meeting	Sample group			
	A1B1	A1B2	A2B1	A2B2
1st	Initial test of spatial ability of Euclid geometry (covariate)			
	Preliminary test of spatial ability of Lobachevsky geometry	Preliminary test of spatial ability of Riemann geometry	Preliminary test of spatial ability of Lobachevsky geometry	Preliminary test of spatial ability of Riemann geometry
2nd	Students learn axiomatic deductive interpreters in geometric systems: concept of base (primitive element), statement of base (axiom), concept of being defined (definition), & statements that must be proved (theorem, lemma, & corollary)			
3rd	Students learn axioms of Euclid's alignment through an ethnomathematics (local culture) approach. They learn to understand importance of learning non-Euclid geometry through neutral geometry. They learn about axiomatic structure of neutral geometry: Archimedes' axiom, Saccheri's quadrilateral concept, & his theorems			
4th	Students learn Lobachevsky geometry system through an ethnomathematics approach: Axiom of Lobachevsky's alignment & some of its logical consequences	Students learn Riemann geometry system through an ethnomathematics approach: Axiom of Riemann's alignment & some of its logical consequences	Students study Lobachevsky geometry systems through conventional approaches: Axioms of Lobachevsky's alignment & some of its logical consequences	Students learn Riemann geometry system through a conventional approach: Axiom of Riemann's alignment & some of its logical consequences
5th	Alignment theorems in Lobachevsky geometry & their proofs	Parallel theorems in Riemann geometry & their proofs	Alignment theorems in Lobachevsky geometry & their proofs	Parallel theorems in Riemann geometry & their proofs
6th	Theorems about triangles in Lobachevsky geometry & their substantiation with local cultural *starting-points*	Theorems about triangles in Riemann geometry & their proofs with local cultural *starting-points*	Theorems about triangles in Lobachevsky geometry & their proofs	Theorems on triangles in Riemann geometry & their proofs
7th	Theorems on quadrilaterals in Lobachevsky geometry & their substantiation with *local cultural starting-points*	Theorems on quadrilaterals in Riemann geometry & their substantiations with *local cultural starting-points*	Theorems about quadrilaterals in Lobachevsky geometry & their proofs	Theorems on quadrilaterals in Riemann geometry & their proofs
8th	Final test of spatial ability of Lobachevsky geometry	Final test of spatial ability of Riemann geometry	Final test of spatial ability of Lobachevsky geometry	Final test of spatial ability of Riemann geometry

material on Riemann geometry; A2B1 is a sample group that is taught using a conventional approach with Lobachevsky geometry material; and A2B2 is the sample group that is taught using a conventional approach with Riemann geometry material. Based on the research design in **Table 1**, details of research activities in the lecture process are summarized in **Table 2**.

Data analysis techniques to achieve goals to achieve goals (4) data are analyzed by covariate analysis (ANCOVA). To achieve goal (3) it is analyzed by frame analysis method and fixed comparison method. Meanwhile, to achieve the objectives (5) an analysis of the implementation of the learning trajectory scenario in the lesson plan (LP) is used.

Data Analysis

The research data were analyzed through a micro-genetic approach (Siegler & Crowley, 1991). The micro genetic approach is an approach to data analysis of learner activity that can generate a more precise description of cognitive changes than is possible. Data analysis used the frame analysis method (FAM) (Karadag, 2009). It was a spiral and cyclic structure with many cycles interacting with each other. Each cycle after the first one interacts with the previous cycle. The collection of mental activities obtained through the frame method is collected in pieces of statement (*tree*) through genetic decomposition analysis to find the characteristics of spatial ability elements of non-Euclid geometry (*forest*) (Karadag, 2009; Widada et al., 2020). The cognitive processes in each of the above stages are always analyzed using the genetic decomposition of students (Widada et al., 2019). Researchers analyzed the data using genetic decomposition analysis techniques. Researchers follow pre-analysis, microanalysis, and tree-to-forest sub-stages for each test item and its arguments. The description of each subject of study is categorized in action-process-object-scheme (Dubinsky & McDonald, 2000). Each stage of mental activity (APOS) can be represented in real activity (Widada, 2011).

The analysis process through a micro-genetic approach with the frame analysis method is carried out based on the following stages. The stages are pre-analysis stage-1, pre-analysis stage-2, pre-analysis stage-3, microanalysis stage-1, microanalysis stage-2,

Table 3. Percentage of learning trajectory scenario implementation

Learning trajectory	Performance percentage (%)					
	LP-1	LP-2	LP-3	LP-4	LP-5	LP-6
Deliver learning objectives	86.50	89.00	90.50	92.00	94.00	97.00
Student exploring	85.50	90.00	90.50	92.50	95.00	97.00
Providing ethnomathematics-based visual problems	92.00	95.50	96.00	96.00	95.00	98.00
Students make conclusions & summaries of exploration results	86.00	89.50	92.00	94.00	97.00	99.00
Students share conclusions about concepts & principles in geometric systems	88.50	91.00	91.50	93.00	95.00	98.00
Mean	**87.70**	**91.00**	**92.10**	**93.50**	**95.20**	**97.80**

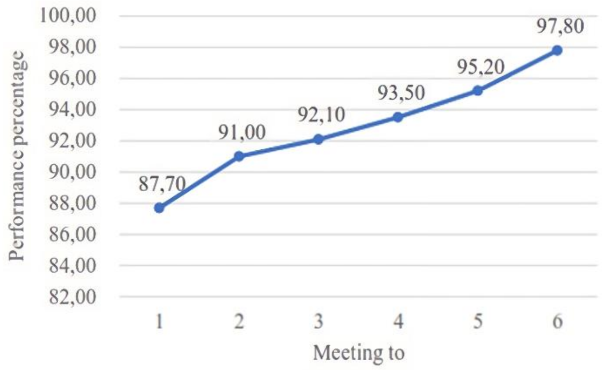

Figure 1. Graph of LP-ethnomathematics implementation (Source: Authors' own elaboration)

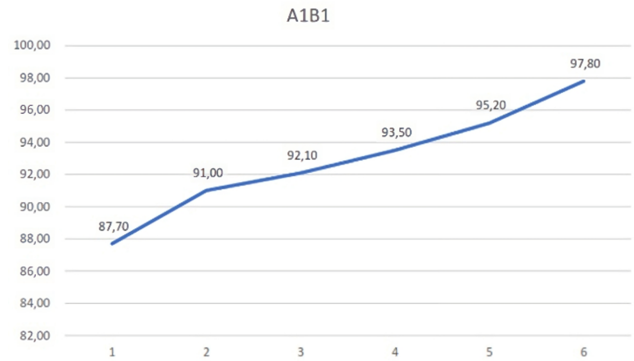

Figure 2. Graph of LP-A1B1 implementation (Source: Authors' own elaboration)

microanalysis stage-3, *tree-to-forest* stage-1, *tree-to-forest* stage-2, and *tree-to-forest* stage-3**.** To obtain a new theory, a process of theoreticization is carried out through a fixed comparison method (Glaser & Strauss, 2006). Experimental data were analyzed using covariate analysis. To facilitate the analysis of the data, the SPSS application program was used.

RESULTS

Non-Euclid geometric spatial ability is the ability to think geometrically through abstraction, idealization and generalization processes based on five elements of mental and physical activity, namely ethnomathematics visualization, mental visualization, iconic relations, symbolic relations, and constructing non-Euclid formal geometry (Nugroho et al., 2022). Those are elements of non-Euclidean geometric spatial abilities with an ethnomathematics approach. The application of this approach to this study was carried out for six lessons in each class. That means that there are twenty-four meetings for all groups. The results of observations at each learning meeting obtained the percentage of implementation of each stage of the learning trajectory.

Spatial learning tracks on non-Euclidean geometry through ethnomathematics learning approach were analyzed on observational data on the implementation of the learning implementation plan (LP). Based on implementation and quasi-experiments, conclusions are obtained about the spatial learning trajectory of non-Euclidean geometry through an ethnomathematics learning approach in terms of APOS theory.

Based on the results of observing the implementation of the learning implementation plan through the ethnomathematics approach, it can be analyzed in such a way that a non-Euclidean geometry spatial learning trajectory is produced through an ethnomathematics learning approach in terms of APOS theory. The results of the observational analysis are listed in **Table 3**.

Based on **Table 3**, it can be seen that the average implementation of LP from LP-1 to LP-6 is more than 85%, which means that each learning trajectory is implemented well, and is increasing. This can be seen from the graph of **Figure 1**.

Based on **Figure 1**, it means that the learning trajectory in learning non-Euclidean geometry with an ethnomathematics approach is well implemented. This is also supported by the results of statistical tests that show that the geometric spatial abilities of students who learn through an ethnomathematics approach are better than conventional student groups.

If viewed based on the groups in the quasi-experiment with a 2x2 factorial design, the implementation of the LP (learning trajectory) can be described, as follows. Each for group A1B1: Lobachevsky geometry and ethnomathematics; group A2B1: Riemann geometry and ethnomathematics; group A1B2: Lobachevsky and conventional geometry; and group A2B2: Riemann and conventional geometry.

Group A1B1: Lobachevsky Geometry and Ethnomathematics

Based on observational data in group A1B1: Lobachevsky geometry and ethnomathematics for six

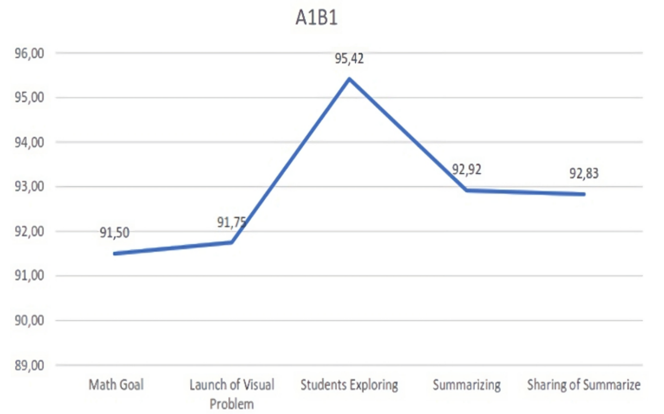

Figure 3. Learning trajectory graph-A1B1 (Source: Authors' own elaboration)

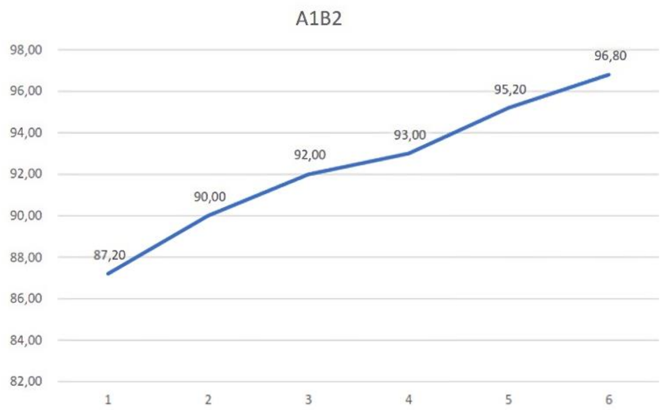

Figure 4. Graph of LP-A1B2 implementation (Source: Authors' own elaboration)

meetings (6 LP), diagram can be presented, as follows. Based on **Figure 2**, implementation of learning trajectory for groups of students studying Lobachevsky geometry with an ethnomathematics approach was implemented 87.70% at LP-1, and continued to increase to 97.80% at meeting six, namely LP-6. This shows that the implementation of LP from 1-6 has reached more than 85%, which means that the learning trajectory for the group of students who study Lobachevsky geometry with an ethnomathematics approach is implemented very well. It can also be seen that the implementation of the learning trajectory for this group of students is very good for every step (**Figure 3**).

Figure 3 shows that each step of the learning trajectory for a group of students studying Lobachevsky geometry with an ethnomathematics approach is accomplished at over 85%. Starting from understanding learning objectives (91.50%), and the highest is student exploration activities (95.42%), to the final step of sharing summaries (92.83%). This shows that the learning trajectory of learning Lobachevsky geometry with an ethnomathematics approach is

(1) delivering learning objectives (learning objective),

(2) providing ethnomathematics-based visual problems,

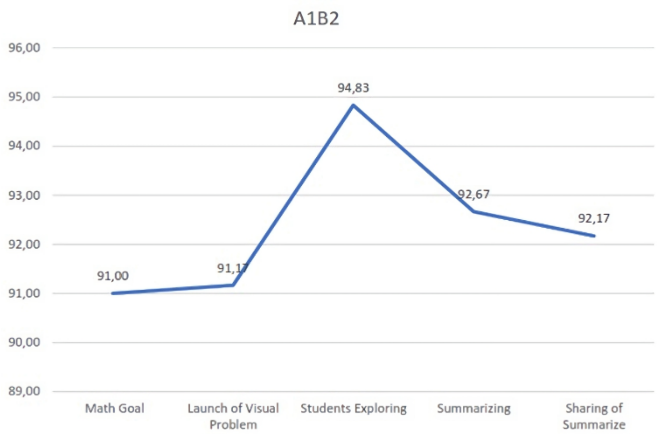

Figure 5. Graph of learning trajectory-A1B2 (Source: Authors' own elaboration)

(3) students do exploration,

(4) students make conclusions and summaries of exploration results, and

(5) students share conclusions/summaries about concepts and principles in geometric systems.

Group A1B2: Lobachevsky and Conventional Geometry

Based on observational data in group A1B2: Lobachevsky and conventional geometry for six meetings (6 LP), diagram in **Figure 4** can be presented.

Based on **Figure 4**, the implementation of the learning trajectory for groups of students studying Lobachevsky geometry with a conventional approach was implemented 87.20% at LP-1, and continued to increase to 96.80% at meeting to6, namely LP-6. This shows that the implementation of LP from 1-6 has reached more than 85%, which means that the learning trajectory for the group of students studying Lobachevsky geometry with the conventional approach is very well implemented. It can also be seen that the implementation of the learning trajectory for this group of students is very good for every step (**Figure 5**).

Figure 5 shows that each step of the learning trajectory for a group of students studying Lobachevsky geometry with a conventional approach is accomplished at over 85%. Starting from understanding learning objectives (91.00%), and the highest is student exploration activities (94.83%), to the final step of sharing summaries (92.17%). This shows that the learning trajectory of learning Lobachevsky geometry with a conventional approach is

(1) delivering learning objectives (learning objective),

(2) providing ethnomathematics-based visual problems,

(3) students do exploration,

(4) students make conclusions and summaries of exploration results, and

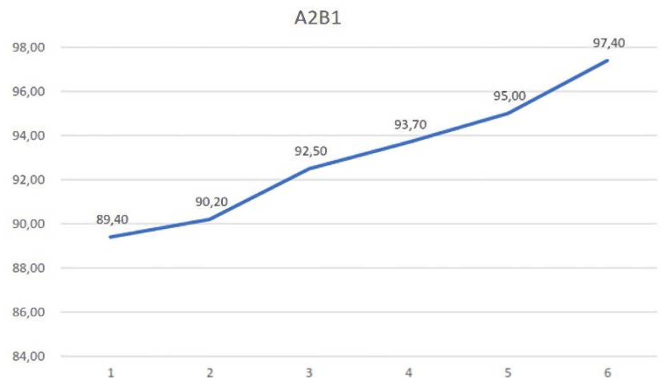

Figure 6. Graph of LP-A2B1 implementation (Source: Authors' own elaboration)

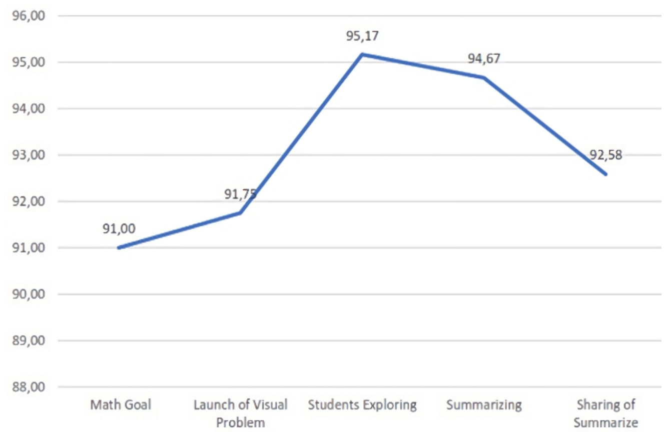

Figure 7. Graph of learning trajectory-A2B1 (Source: Authors' own elaboration)

(5) students share conclusions/summaries about concepts and principles in geometric systems.

Group A2B1: Riemann Geometry and Ethnomathematics

Based on observational data in group A2B1: Riemann geometry and ethnomathematics for six meetings (6 LP), the diagram in **Figure 6** can be presented.

Based on **Figure 6**, implementation of learning trajectory for groups of students studying Riemann geometry with an ethnomathematics approach was implemented 89.40% at LP-1, and continued to increase to 97.40% at meeting to6, namely LP-6. This shows that implementation of LP from 1-6 has reached more than 85%, which means that learning trajectory for groups of students who study Riemann geometry with an ethnomathematics approach are implemented very well. It can be seen that implementation of learning trajectory for students is very good for every step (**Figure 7**).

Figure 7 shows that each step of the learning trajectory for a group of students studying Riemann geometry with an ethnomathematics approach accomplished above 85%. Starting from understanding learning objectives (91.00%), and the highest is student exploration activities (95.17%), to the final step of sharing summaries (92.59%). This shows that the

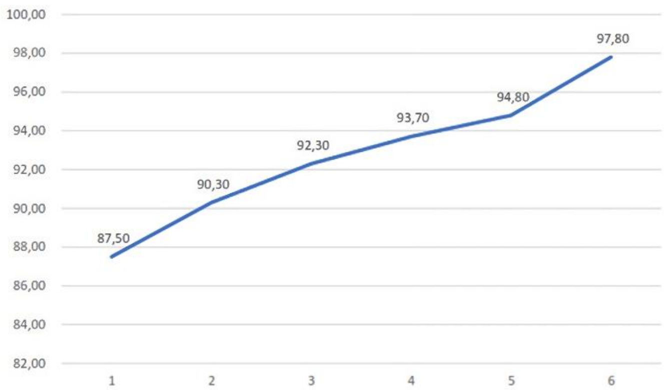

Figure 8. Graph of LP-A2B2 implementation (Source: Authors' own elaboration)

learning trajectory of learning Riemann geometry with an ethnomathematics approach is

(1) delivering learning objectives (learning objective),

(2) providing ethnomathematics-based visual problems,

(3) students do exploration,

(4) students make conclusions and summaries of exploration results, and

(5) students share conclusions/summaries about concepts and principles in geometric systems.

Group A2B2: Riemann and Conventional Geometry

Based on observational data in group A1B2: Riemann and conventional geometry for six meetings (6 LP), the diagram in **Figure 8** can be presented.

Based on **Figure 8**, the implementation of the learning trajectory for groups of students studying Riemann geometry with a conventional approach was implemented 87.50% at LP-1, and continued to increase to 97.80% at meeting to6, namely LP-6. This shows that the implementation of LP from 1-6 has reached more than 85%, which means that the learning trajectory for groups of students who study Riemann geometry with the conventional approach are implemented very well. It can also be seen that the implementation of the learning trajectory for this group of students is very good for every step (**Figure 9**).

Figure 9 shows that each step of learning trajectory for groups of students studying Riemann geometry with the conventional approach is accomplished at over 85%. Starting from understanding learning objectives (91.67%), and the highest is student exploration activities (95.25%), to the final step of sharing summaries (92.50%). This shows that the learning trajectory of learning Riemann geometry with a conventional approach is

(1) delivering learning objectives (learning objective),

(2) providing ethnomathematics-based visual problems,

(3) students do exploration,

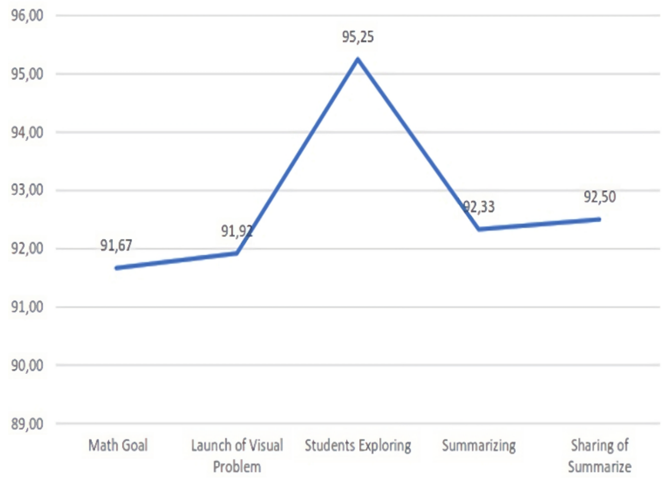

Figure 9. Graph of learning trajectory-A2B2 (Source: Authors' own elaboration)

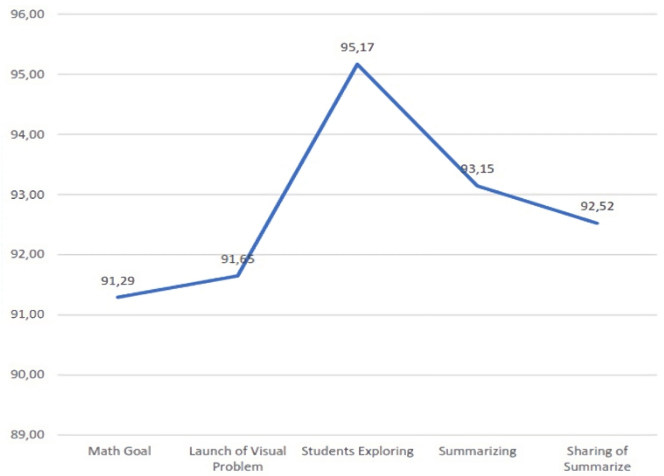

Figure 10. Average of each learning step of 4 sample groups (Source: Authors' own elaboration)

(4) students make conclusions and summaries of exploration results, and

(5) students share conclusions/summaries about concepts and principles in geometric systems.

Based on the explanation above, it can be seen that the learning trajectory is consistently implemented very well. This can also be seen from the overall data analysis listed in the graph, as follows.

Based on **Figure 10**, it is certain that the learning trajectory of learning non-Euclid geometry (Lobachevsky and Riemann) is

(1) delivering learning objectives (learning objectives) (91.29%),

(2) providing ethnomathematics-based visual problems (91.65%),

(3) students do exploration (95.17%),

(4) students make conclusions and summaries of exploration results (93.15%), and

(5) students share conclusions/summaries about concepts and principles in geometric systems (92.62%).

Based on this research, which has been published, the application of the ethnomathematics approach to learning non-Euclidean geometry improves the spatial abilities of non-Euclidean geometry (Nugroho et al., 2022). That is data analysis using covariate analysis (ANCOVA). In this publication it was concluded that there were differences in spatial ability between students with Lobachevsky geometry and Riemann geometry after controlling for the influence of Euclid's spatial ability in geometry. There is a difference in spatial abilities between students who were given an ethnomathematics and conventional approach after controlling for the effect of Euclid geometry's spatial ability. There is an interaction effect of geometry material and learning approach on the spatial ability of non-Euclid geometry after controlling for the influence of the spatial ability of Euclidean geometry. There is a linear effect of covariate Euclidean geometry spatial ability on non-Euclid geometry spatial ability. Geometry spatial ability, geometry material and Euclid's learning approach together influence spatial ability. The students' spatial ability for Lobachevsky geometry material is not higher than the Riemann geometry material, which is taught with an ethnomathematics learning approach after controlling for the effect of Euclid's geometry spatial ability. The students' spatial abilities for Lobachevsky geometry are not higher than the material for Riemann geometry, which is taught with the conventional learning approach after controlling for the influence of the spatial abilities of Euclid geometry. The spatial ability of students who were given the ethnomathematics learning approach was higher than students who were given the conventional learning approach for Lobachevsky geometry after controlling for the influence of Euclid's geometry spatial abilities. The spatial ability of the students who were given the ethnomathematics learning approach was higher than the students who were given the conventional learning approach for Riemann geometry after controlling for the influence of the spatial abilities of Euclid's geometry. Based on the results of this study, the learning trajectory is the right learning trajectory for learning non-Euclidean geometry through an ethnomathematics approach.

DISCUSSION

Based on this research, which has been published, the application of the ethnomathematics approach to learning non-Euclidean geometry improves the spatial abilities of non-Euclidean geometry (Nugroho et al., 2022). That is data analysis using covariate analysis (ANCOVA). In this publication it was concluded that there were differences in spatial ability between students with Lobachevsky geometry and Riemann geometry after controlling for the influence of Euclid's spatial ability in geometry. There is a difference in spatial abilities between students who were given an ethnomathematics and conventional approach after controlling for the effect of Euclid geometry's spatial ability. There is an interaction effect of geometry material and learning approach on the spatial ability of non-Euclid geometry after controlling for the influence of the spatial ability of Euclidean geometry. There is a linear effect of covariate Euclidean geometry spatial ability on non-Euclid geometry spatial ability. Geometry spatial ability, geometry material and Euclid's learning approach together influence spatial ability. The students' spatial ability for Lobachevsky geometry material is not higher than the Riemann geometry material, which is taught with an ethnomathematics learning approach after controlling for the effect of Euclid's geometry spatial ability. The students' spatial abilities for Lobachevsky geometry are not higher than the material for Riemann geometry, which is taught with the conventional learning approach after controlling for the influence of the spatial abilities of Euclid geometry. The spatial ability of students who were given the ethnomathematics learning approach was higher than students who were given the conventional learning approach for Lobachevsky geometry after controlling for the influence of Euclid's geometry spatial abilities. The spatial ability of the students who were given the ethnomathematics learning approach was higher than the students who were given the conventional learning approach for Riemann geometry after controlling for the influence of the spatial abilities of Euclid's geometry.

Based on the results of this study, the learning trajectory is the right learning trajectory for learning non-Euclidean geometry through an ethnomathematics approach. Those are the five steps of the learning trajectory, as follows.

Step 1: Math Goal (Learning Objective)

Lecturers and students review the learning objectives of geometry based on visual problems with an ethnomathematics approach. Lobachevsky's geometry uses ethnomathematics: bubu, trumpet, and goa tourism. Riemann geometry: grapefruit, sphere, and globe.

Step 2: Launch of Visual Problems

In step 2, the lecturer launched a visualization-based problem with an ethnomathematics approach for the class as a whole. Through student activity sheets, lecturers help students understand problem settings, mathematical contexts, and challenges. The following questions can help lecturers prepare for launching: estimate what students will do? What do students need to understand the context of the story and the challenges of the problem? What difficulties can be predicted for students? How can you prevent giving help too far from the given problem? The launch phase is also a time for lecturers to introduce new ideas, clarify definitions, review old concepts, and relate problems to previous student experiences. Lecturers must be careful not to lecture too much, and give too little challenge from routine assignments, or cut off the structure of the strategy from an open launch of the problem.

Step 3: Student Exploring

In the explore phase, students work to solve problems individually, in pairs, in small groups, or sometimes in a class as a whole. Their work, such as collecting data, sharing ideas, making patterns, making conjectures, and developing problem solving strategies. The role of the lecturer in this phase is to walk around the class, observe student performance individually, and encourage students to carry out assignments. Lecturers help students to work diligently by asking questions and confirming what is needed. For students who are interested and adept at in-depth investigations, lecturers provide extra challenges related to problems. The following questions can help lecturers prepare for the exploring phase:

(1) How will the lecturer organize students to explore this problem? (individually?, in pairs?, groups?, class as a whole?),

(2) What materials will students need?,

(3) How will students record and report their work?,

(4) Anticipate what can be done to deal with the various strategies they use?,

(5) What questions can be given to encourage students to keep working, thinking and studying?,

(6) What questions can be given to focus their thinking if they are frustrated or stop carrying out tasks?, and

(7) What questions can be given to challenge students if the initial level questions are "answered"?

Step 4: Student of Summarize

The summary phase begins when most students have collected sufficient data or have made sufficient progress toward solving the problem. In this phase, students discuss their solutions, as well as the strategies they use to approach problems, organize data, and find solutions. Through discussions, lecturers help students improve their understanding of mathematics in problems and guide them in improving their strategies so that problem-solving techniques are efficient and effective. In the summary discussion gives instructions so that lecturers and students play a significant role. Ideally, they will ask conjectures, ask questions, try alternatives, reason, refine their strategies and conjectures, and make connections.

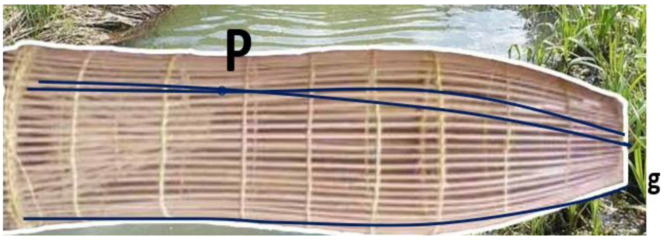

Figure 11. Cognitive process based on *bubu* (Source: Authors' own elaboration)

As a result of the discussion, students will become more skilled at using ideas and techniques that generate experience with the problems they face. In the summarizing phase, it contains problems, investigations, and units that are intended so that the lecturer can estimate the level of development of students' mathematical knowledge. The following questions can help the teacher prepare a summary:

(1) How can lecturers help students understand and appreciate the various methods used?,

(2) How can lecturers arrange discussion procedures about student summaries in thinking about problems?,

(3) What concepts and strategies are needed to make conclusions?,

(4) What definition or strategy do we need to generalize?,

(5) What linkages and extensions can be made?,

(6) What new questions have increased and how to handle them?, and

(7) What are the possible follow-ups, practices, or implementation of ideas after the summary?

Step 5: Share of Summarize

In this step, students share a summary. Lecturers give instructions for students to play a significant role. Ideally, they will ask conjectures, ask questions, alternative attempts, reason, refine their strategies and conjectures, and make connections. As a result of the discussion, students will become more skilled at using ideas and techniques that generate experience with the problems they face.

The following is one application of the learning trajectory (HLT) in Lobachevsky geometry with an ethnomathematics approach to traditional fishing gear (bubu). This is a student's cognitive process in terms of APOS theory (student genetic decomposition) (Dubinsky & McDonald, 2000).

Action

To facilitate students to take "action" namely:

(1) understanding real problems related to Lobachevsky's parallel axiom, namely "bubu". (Bengkulu people's traditional fishing gear).

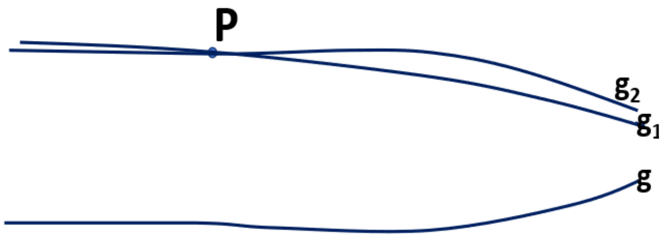

Figure 12. Cognitive process to achieve a principle (Source: Authors' own elaboration)

(2) choose a binding site for each pot stick,

(3) one such place is named point P,

(4) choose a stick at the bottom of the pot that is not tied at point P,

(5) name one stick with the line g,

(6) choose two sticks tied to the bond at point P, and

(7) name the two sticks with the lines g1 and g2.

Mental and physical activities based on real-world problems regarding bubu ethnomathematics, can be described, as follows (**Figure 11**).

It is a cognitive process in the form of mathematical connections: A student who succeeds in making mathematical connections, namely connecting how geometric ideas are related to bubu; relates new problems to old ones by asking, "where have I seen problems like this before?"; likes seeing how Lobachevsky geometry ideas or concepts connect to the real world; can easily relate familiar ideas to concepts or new skills and love to know when others are thinking about a solution strategy in a different way.

Process

Process is contemplated action. A process can be obtained by performing an action repeatedly. This stage is marked when someone can think of doing a process without actually doing it and can think about how to reverse or arrange a process. To facilitate students to carry out the "process (process)", namely:

(1) represents one bond point with one point P,

(2) represents a stick at the bottom of the pot that is not tied at point P is a line g, and

(3) represents two sticks tied to bond at point P with two lines, namely the lines g1 and g2 (**Figure 12**).

This is the cognitive process of mathematical representation (a student who succeeds in representation) i.e., having a list of ways to represent a problem and its solution; uses a series of representations in expressing his thoughts, (words, pictures or drawings, graphics or other graphics); uses representation to reveal what he thinks, how he knows it, and how the problem is solved; can easily move from one type of representation to another and know the right or proper representation to use and when to use it.

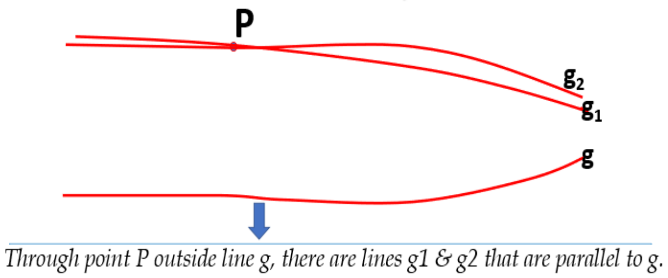

Figure 13. Achievement of Lobachevsky's parallel principle (Source: Authors' own elaboration)

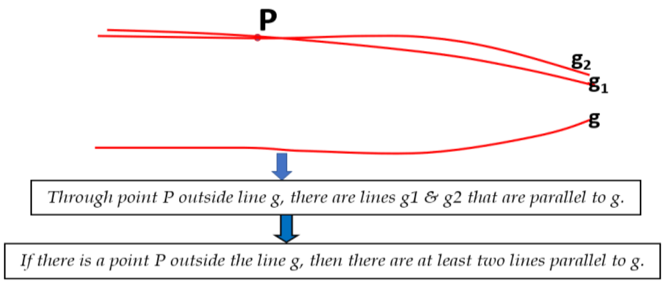

Figure 14. Achievement of Lobachevsky's parallel principle (Source: Authors' own elaboration)

Object

Object is a totality that is carried out in a process. This stage is marked by a person's ability to act on the object and provide reasons or explanations about its properties and can perform re-decomposition. To facilitate students to do "objects", this module contains:

(1) questions that consist of several concepts,

(2) questions that encourage students to provide an explanation of what they wrote,

(3) questions that encourage students to re-describe the properties of a concept, and

(4) questions that train students to be able to flip through concepts in the material being taught.

In order to obtain the principle of parallel axioms of Lobachevsky geometry as outlined in **Figure 13**.

It is a cognitive process in the form of mathematical communication (a student who successfully communicates mathematically about Lobachevsky's alignment). Students are able to explain their thoughts clearly and concisely, namely seeking clarification about two lines that are parallel to a certain line; attempted to make a new statement about the axiom of equality in special cases; provide an explanation or try to figure out why the statement makes sense.

Scheme

A schema for a particular Lobachevsky geometry concept is a collection of actions, processes and objects connected by some general principles so as to form an interrelated framework in one's mind. To facilitate students to carry out "schemas", namely, to include achieving a statement in the form of the Lobachevsky parallel axiom as a result of an assignment that relates general situations involving several concepts being studied (**Figure 14**).

This process illustrates that students are able to carry out problem solving processes well, through an ethnomathematics approach. This means that a student is a successful problem solver, that is showing confidence in solving Lobachevsky's parallel problems through real media; show persistence when faced with difficult problems and do not give up; when given an unfamiliar problem, knows what to do and can switch strategies if one does not work; and has a list of problem-solving strategies to invoke when solving problems about Lobachevsky's parallel axiom.

These results provide support for previous research. Setiadi et al. (2019) stated that the trajectory of learning three-dimensional material through the Euclidean geometry approach can overcome learning obstacles that occur in studying this topic through the Euclidean geometry approach. The stages of the creative thinking process that students have are orientation, preparation, incubation, illumination and verification, which will be passed as a point of student thinking. That is through the trajectory of spatial thinking (Arnis et al., 2019). The developmental trajectory of children's spatial skills influences variables and associations with subsequent mathematical thinking (Möhring et al., 2021). It is the influence of various covariates of language skills on predicting a person's spatial development.

Early spatial reasoning predicts later mathematical understanding, indicating that early spatial reasoning may play an important role in learning mathematics. Yuliardi and Rosjanuardi (2021) state that spatial abilities and the development of spatial ability theory are related to spatial conceptions in students' understanding.

HLT design consists of three phases: initial design, experiments, and retrospective analysis. HLT results are then refined into LIT (local instructional trajectory). The conclusion of the research is that HLT is very important for teachers to develop learning trajectories as a reference in designing learning that can optimize spatial abilities. Spatial abilities are needed by students to learn the concept of geometry,

Thus, the learning trajectory of non-Euclidean geometry spatial learning through an ethnomathematics learning approach in terms of APOS theory is, as follows: delivering learning objectives (learning objective); providing ethnomathematics-based visual problems; students do exploration; students make conclusions and summaries of exploration results; and ends with students sharing conclusions/summaries about concepts and principles in geometric systems.

CONCLUSIONS

Learning Euclid and non-Euclid geometry through ethnomathematics approaches has a positive impact on students' spatial abilities. There was an increase in the spatial ability of students in non-Euclidean geometry after learning through an ethnomathematics approach. It means that the spatial ability of the students who were given the ethnomathematics learning approach was higher than the students who were given the conventional learning approach for Lobachevsky geometry material after controlling for the influence of Euclidean geometry spatial abilities.

Also, the same thing happened for the spatial abilities of Riemann geometry students. It is obtained based on the application of the non-Euclidean geometry spatial learning trajectory, namely conveying learning objectives (learning objective); providing ethnomathematics-based visual problems; students do exploration; students make conclusions and summaries of exploration results; and ends with students sharing conclusions/summaries about concepts and principles in geometric systems.

Author contributions: YLS, KUZN, SS, & BW: reviews & editing; **YLS, SS, & BW:** fundraising; **YLS & KUZN:** conceptualization, data collection, & writing original draft; **KUZN & SS:** methodology; **KUZN, SS, & BW:** interview; & **KUZN & BW:** validation, data analysis, & administration. All authors have agreed with the results and conclusions.

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Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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